THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050B Mathematical Analysis I
Tutorial 4
Date: 3 October, 2024

1. Let $\{I_n = [a_n, b_n]\}_{n=1}^{\infty}$ be a sequence of nested closed bounded intervals and let $\zeta = \sup\{a_n : n \in \mathbb{N}\}$ and $\eta = \inf\{b_n : n \in \mathbb{N}\}$. Show that $\eta \in \bigcap_{n=1}^{\infty} I_n$, and $[\zeta, \eta] = \bigcap_{n=1}^{\infty} I_n$.

2. Use the $\varepsilon - N$ definition of limit to show

   (a) $\lim_{n\to\infty} \dfrac{n^2 - 1}{2n^2 + 3} = \dfrac{1}{2}$,

   (b) $\lim_{n\to\infty} \sqrt{n^2 + 1} - n = 0$,

   (c) $\lim_{n\to\infty} (2n)^{\frac{1}{n}} = 1$,

3. Show that $\{(-1)^n\}_{n=1}^{\infty}$ does not converge.

Announcement: HW2 posted on website, due 10/10 2359 on gradescope

1. Let $\{I_n = [a_n, b_n]\}_{n=1}^{\infty}$ be a sequence of nested closed bounded intervals and let $\zeta = \sup\{a_n : n \in \mathbb{N}\}$ and $\eta = \inf\{b_n : n \in \mathbb{N}\}$. Show that $\eta \in \bigcap_{n=1}^{\infty} I_n$, and $[\zeta, \eta] = \bigcap_{n=1}^{\infty} I_n$.

Pf: First, WTS $\eta \in \bigcap_{n=1}^{\infty} I_n \iff \eta \in I_n$ for all n.

Clearly by infimum, $\eta \le b_n$ for each n. So it remains to show $a_n \le \eta$ for all n

We will show that each a_n is a lower bdd. of $\{b_k : k \in \mathbb{N}\}$

2 Cases:

1) $n \le k$ Then $I_k \subseteq I_n$ so $a_n \le a_k \le b_k \le b_n$ ✓

2) $k < n$ Then $I_n \subseteq I_k$, so $a_k \le a_n \le b_n \le b_k$ ✓

So each a_n is a l.b. of the set $\{b_k : k \in \mathbb{N}\}$. So by infimum, $a_n \le \eta$ for all n.

So $\eta \in \bigcap_{n=1}^{\infty} I_n$.

Similarly, we can show $\zeta \in \bigcap_{n=1}^{\infty} I_n \Rightarrow [\zeta, \eta] \subseteq \bigcap_{n=1}^{\infty} I_n$

Remains to show $\bigcap_{n=1}^{\infty} I_n \subseteq [\zeta, \eta]$. Let $z \in \bigcap_{n=1}^{\infty} I_n$ That means $a_n \le z \le b_n$ for all n.

In particular, z is an u.b. of the set $\{a_n : n \in \mathbb{N}\}$

So by supremum, we have $\zeta \le z$

Similarly, z is a l.b. of the set $\{b_n : n \in \mathbb{N}\}$. so by infimum, we have $z \le \eta$. So $z \in [\zeta, \eta]$

//.

2. Use the $\varepsilon - N$ definition of limit to show

(a) $\lim_{n\to\infty} \frac{n^2-1}{2n^2+3} = \cancel{0}, \frac{1}{2}$

(b) $\lim_{n\to\infty} \sqrt{n^2+1} - n = 0,$

(c) $\lim_{n\to\infty} (2n)^{\frac{1}{n}} = 1,$

Rationalize: $\left|\sqrt{n^2+n} - n\right| = \left|\left(\sqrt{n^2+1} - n\right)\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}\right|$

Recall: a sequence $\{x_n\}$ converges to $L \in \mathbb{R}$ if for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ s.t. for all $n \geq N$ we have

$$|x_n - L| < \varepsilon.$$

1) let $\varepsilon > 0$ be given.
2) Guess the limit L
3) Estimate $|x_n - L|$ for each n
4) Choose a N that works.

Pf: a) let $\varepsilon > 0$ be given.

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|\frac{2n^2-2}{4n^2+6} - \frac{2n^2+3}{4n^2+6}\right|$$

$$= \left|\frac{-5}{4n^2+6}\right|$$

$4n^2+6 \geq 4n^2$

$\Downarrow$

$\frac{1}{4n^2+6} \leq \frac{1}{4n^2}$

$$\leq \frac{5}{4n^2}$$

Chosing $N(\varepsilon) > \sqrt{\frac{5}{4\varepsilon}}$, then for all $n \geq N$, we have

A.P.

$$\left|x_n - \frac{1}{2}\right| \leq \frac{5}{4n^2} < \frac{5}{4\left(\sqrt{\frac{5}{4\varepsilon}}\right)^2} = \frac{5}{4 \cdot \frac{5}{4\varepsilon}} = \varepsilon$$

Note: I could have also chosen $\frac{5}{4n^2} < \frac{5}{4n}$ and then take $N(\varepsilon) > \frac{5}{4\varepsilon}$.

c) Let $\varepsilon > 0$ be given. For $n \geq 1$, note that $(2n)^{\frac{1}{n}} > 1$, so for each n, we can write $(2n)^{\frac{1}{n}} = 1 + k_n$ for some k_n. Since the limit is 1, if we show that $k_n \to 0$ as $n \to \infty$, then we have

$$\left|(2n)^{\frac{1}{n}} - 1\right| = \left|1 + k_n - 1\right| = |k_n| \to 0.$$

We have $2n = (1+k_n)^n = 1 + nk_n + \frac{1}{2}n(n-1)k_n^2 + \cdots$
$$\geq \frac{1}{2}n(n-1)k_n^2$$

So then rearranging we have

$$k_n^2 \leq \frac{4n}{n(n-1)} = \frac{4}{n-1}$$

So $|k_n - 0| \leq \frac{4}{n-1}$. So taking $N(\varepsilon) > \frac{4}{\varepsilon} + 1$, we have, for all $n \geq N$,

$$|k_n - 0| \leq \frac{4}{n-1} < \frac{4}{\frac{4}{\varepsilon} + 1 - 1} = \varepsilon$$

3. Show that $\{(-1)^n\}_{n=1}^{\infty}$ does not converge.

Out of time (left to next week): Preview:

Consider $n=2k$, $n=2k+1$.